MAT 126 Solutions to Final Exam

1. Evaluate the following integrals.
   If you find an integral to be divergent, write DIVERGENT.

10 pts (a)

$$\int x^2 e^x \, dx$$

Solution: If we use integration by parts, we'll be able to reduce the power of the x-term until it's gone:

$$\begin{aligned}\int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx \\ &= x^2 e^x - 2\left(x e^x - \int e^x \, dx\right) \\ &= x^2 e^x - 2x e^x + 2\int e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + c \\ &= e^x(x^2 - 2x + 2) + c.\end{aligned}$$

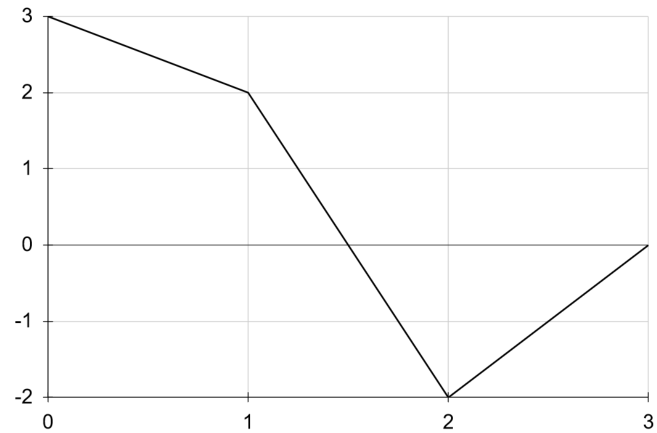
10 pts (b)

$$\int_1^\infty \frac{1}{t^3} \, dt$$

Solution: Since this is an improper integral, we must proceed by taking a limit:

$$\begin{aligned}\int_1^\infty \frac{1}{t^3} \, dt = \lim_{a\to\infty}\left(\int_1^a t^{-3}\right) &= \lim_{a\to\infty}\left(\left[-\frac{1}{2}t^{-2}\right]_1^a\right) \\ = \lim_{a\to\infty}\left(-\frac{1}{2a^2} + \frac{1}{2}\right) &= \frac{1}{2}.\end{aligned}$$

2. The graph of $y = f(x)$ is shown below.



Let $g(x) = \int_0^x f(t)\,dt$. What are the values of the following?

6 pts (a) $g(1)$

Solution: By the Fundamental Theorem of Calculus, $g(1)$ is given by the (signed) area under the curve $y = f(x)$ between $x = 0$ and $x = 1$. From the graph, we compute the area of a square and a triangle to get

$$g(1) = 2 + \frac{1}{2} = \frac{5}{2}.$$

6 pts (b) $g(3)$

Solution: The same logic applies as in part (a), but this time we must remember that we are counting up the *signed* area under the curve. The area under the curve between $x = 1$ and $x = 3/2$ cancels out with the area between $x = 3/2$ and $x = 2$, so that

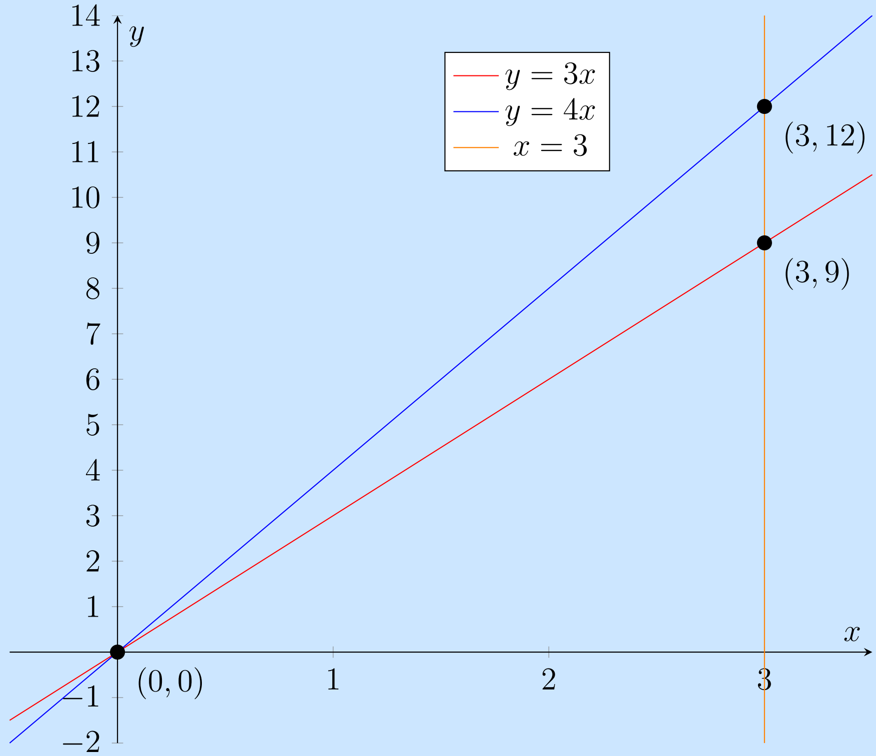
$$g(3) = \frac{5}{2} - 2 \times \frac{1}{2} = \frac{3}{2}.$$

8 pts (c) $g'(2)$

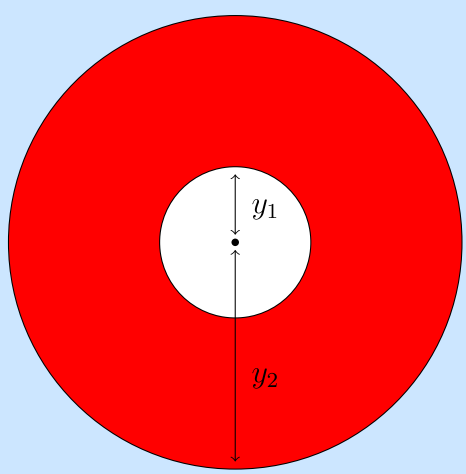
Solution: By the (other half of the) Fundamental Theorem of Calculus, $g'(x) = f(x)$, so that $g'(2) = f(2) = -2$.

20 pts 3. Consider the region enclosed by the lines $y = 3x, y = 4x$ and $x = 3$. What is the volume of the solid obtained by rotating this region about the x-axis?

Solution: Our first step is to graph the region in question. We have two lines through the origin of different steepness, and a vertical line at $x = 3$.



After rotating the region about $y = 0$ (the x-axis), a typical cross-section cut parallel to the y-axis (say, at $x = x_0$) looks like this:



where $y_1 = 3x_0$ is the distance from $(x_0, 0)$ to $(x_0, 3x_0)$, and $y_2 = 4x_0$ is the distance from $(x_0, 0)$ to $(x_0, 4x_0)$. It follows that, in general, the area of an arbitrary cross-section is given by

$$\begin{aligned} A(x) &= \pi(y_2(x)^2 - y_1(x)^2) = \pi((4x)^2 - (3x)^2) \\ &= \pi(16x^2 - 9x^2) = 7\pi x^2. \end{aligned}$$

Finally, we integrate over the x-values in the region to get the solid's volume.

$$\begin{aligned} V &= \int_0^3 A(x)\,dx = 7\pi \int_0^3 x^2\,dx \\ &= 7\pi \left[\frac{x^3}{3}\right]_0^3 = 7\pi\left(\frac{3^3}{3}\right) = 63\pi. \end{aligned}$$

20 pts

4. Let $f(x)$ be an increasing function on the interval $[0, 1]$. Rank the following in order, from smallest value to largest value:
   - i) $f(0)$
   - ii) $f(1)$
   - iii) The approximation of $\int_0^1 f(x)\,dx$ using left end-points and $n = 5$.
   - iv) The approximation of $\int_0^1 f(x)\,dx$ using right end-points and $n = 5$.
   - v) The average value of $f(x)$ on $[0, 1]$.

Solution: We begin with some observations. Let I_L and I_R be the integral approximations using left and right endpoints, respectively. Since $f(x)$ is increasing, left endpoints will give an underestimate and right endpoints will give an overestimate, so

$$I_L < \int_0^1 f(x)\,dx < I_R.$$

Next, recall that the average value of an arbitrary function $g(x)$ on an interval $[a, b]$ is given by

$$g_{av} = \frac{\int_a^b g(x)\,dx}{b-a}.$$

In particular, $f_{av} = \int_0^1 f(x)\,dx$, so that we have

(iii) < (v) < (iv).

Finally, note that $f(0)$ is simply the approximation of $\int_0^1 f(x)\,dx$ using left endpoints with $n = 1$. Since we are using less endpoints (compared to $n = 5$), the underestimate will be less accurate, so that (i) $<$ (iii).

The same logic applies to $f(1)$ being a worse overestimate, so that (iv) $<$ (ii). Putting everything together, we have

$$\text{(i)} < \text{(iii)} < \text{(v)} < \text{(iv)} < \text{(ii)}.$$

5. Consider the curve $y = x^2$.

10 pts (a) Find functions $f(t), g(t)$ such that the curve is described parametrically by

$$x = f(t),$$
$$y = g(t).$$

Solution: We may choose any functions f, g so long as $g(t) = f(t)^2$ (and $f(t)$ has the entire real line as its *image* - this just means that we can achieve any value we want for $f(t)$ by choosing the right value of t, and is required because x has to be able to range across the entire x-axis).

For simplicity's sake, let's choose $f(t) = t$. Then we must have $g(t) = f(t)^2 = t^2$.

10 pts (b) Write an integral that gives the length of this curve between the points $(0, 0)$ and $(2, 4)$. [You do not have to evaluate this integral.]

Solution: This is a simple application of our *arc length formula*, for which we need the derivatives of $f(t)$ and $g(t)$. We have

$$f'(t) = 1, \text{ and}$$
$$g'(t) = 2t.$$

Our formula then gives us the required integral as

$$\int_0^2 \sqrt{1^2 + (2t)^2}\,dt = \int_0^2 \sqrt{4t^2 + 1}\,dt.$$

(Note that, upon factoring 4 out of the root, we end up with the integral from Question 6.)

6. Consider the integral $\int \sqrt{x^2 + \frac{1}{4}}\, dx$.

6 pts

(a) Using the substitution $x = \frac{1}{2}\tan\theta$, show that

$$\int \sqrt{x^2 + \frac{1}{4}}\, dx = \frac{1}{4}\int \sec^3\theta\, d\theta.$$

Solution: We note that $\tan^2\theta + 1 = \sec^2\theta$, and $\frac{dx}{d\theta} = \frac{1}{2}\sec^2(\theta)$. Thus,

$$\begin{aligned}
\int \sqrt{x^2 + \frac{1}{4}}\, dx &= \int \sqrt{\frac{1}{4}\tan^2(\theta) + \frac{1}{4}}\left(\frac{1}{2}\sec^2(\theta)\, d\theta\right) \\
&= \frac{1}{2}\int \sec^2(\theta)\sqrt{\frac{1}{4}(\tan^2(\theta) + 1)}\, d\theta \\
&= \frac{1}{2}\int \frac{1}{2}\sec^2(\theta)\sqrt{\tan^2(\theta) + 1}\, d\theta \\
&= \frac{1}{4}\int \sec^2(\theta)\sqrt{\sec^2(\theta)}\, d\theta \\
&= \frac{1}{4}\int \sec^3\theta\, d\theta.
\end{aligned}$$

10 pts

(b) Using integration by parts, show that

$$2\int \sec^3\theta\, d\theta = \tan\theta\sec\theta + \int \sec\theta\, d\theta.$$

[Hint: Consider $\int \sec^3\theta\, d\theta$ for the integration by parts, and recall that $\frac{d}{d\theta}(\sec\theta) = \tan\theta\sec\theta$.]

Solution: Integration by parts with $u = \sec\theta$ and $v' = \sec^2\theta$ gives

$$\begin{aligned}
\int \sec^3\theta\, d\theta &= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\, d\theta \\
&= \tan\theta\sec\theta - \int \left(\sec^2\theta - 1\right)\sec\theta\, d\theta \\
&= \tan\theta\sec\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta.
\end{aligned}$$

Rearranging by adding $\int \sec^3\theta\, d\theta$ to both sides gives the answer.

6 pts

(c) Find $\frac{d}{d\theta}\left(\sec\theta + \tan\theta\right)$.

Solution:

$$\begin{aligned}\frac{d}{d\theta}\left(\sec\theta + \tan\theta\right) &= \frac{d}{d\theta}\left(\sec\theta\right) + \frac{d}{d\theta}\left(\tan\theta\right) \\ &= \tan\theta\sec\theta + \sec^2\theta.\end{aligned}$$

10 pts

(d) By writing $\sec\theta$ as

$$\sec\theta = \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta},$$

show that

$$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c.$$

Solution: We proceed as the question suggests:

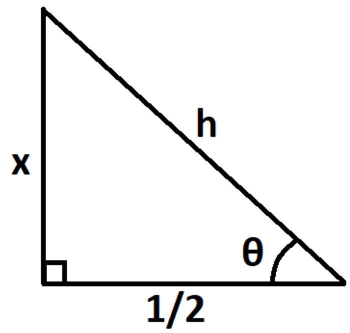
$$\begin{aligned}\int \sec\theta\, d\theta &= \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\, d\theta \\ &= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \\ &= \int \frac{\frac{d}{d\theta}\left(\sec\theta + \tan\theta\right)}{\sec\theta + \tan\theta}\, d\theta.\end{aligned}$$

At this point, we can either use the substitution $u = \sec\theta + \tan\theta$, or (equivalently) we can recall that

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c,$$

which immediately gives the result.

4 pts (e) Consider the below triangle.



Find the values of $h, \tan\theta, \cos\theta$, and $\sec\theta$ (in terms of x).

[Do NOT physically measure the triangle; it is not intended to be drawn to scale.]

Solution: By Pythagoras' Theorem,

$$h = \sqrt{x^2 + (1/2)^2} = \sqrt{x^2 + 1/4}.$$

The 'SOH CAH TOA' trigonometry laws then give us

$$\tan\theta = \frac{\text{O}}{\text{A}} = \frac{x}{1/2} = 2x$$

and

$$\cos\theta = \frac{\text{A}}{\text{H}} = \frac{1/2}{h} = \frac{1}{2\sqrt{x^2 + 1/4}}.$$

Finally, recall that $\sec\theta = 1/\cos\theta$, so that

$$\sec\theta = 2\sqrt{x^2 + 1/4}.$$

4 pts

(f) Evaluate the integral $\int \sqrt{x^2 + \frac{1}{4}}\, dx$.

(i.e. find a function $F(x)$ such that $F'(x) = \sqrt{x^2 + \frac{1}{4}}$.)

Solution: This is simply a matter of putting together parts (a)-(e):

$$\begin{aligned}
\int \sqrt{x^2 + 1/4}\, dx &= \frac{1}{4} \int \sec^3 \theta \, d\theta && \text{(by part a)} \\
&= \frac{1}{8} \left(\tan\theta \sec\theta + \int \sec\theta \, d\theta \right) && \text{(by part b)} \\
&= \frac{1}{8} \tan\theta \sec\theta + \frac{1}{8} \ln|\sec\theta + \tan\theta| + c && \text{(by part d)} \\
&= \frac{1}{2} x \sqrt{x^2 + 1/4} + \frac{1}{8} \ln|2\sqrt{x^2 + 1/4} + 2x| + c && \text{(by part e).}
\end{aligned}$$

As an aside, in conjunction with Question 5, we are now able to compute the arc length of $y = x^2$ between $(0, 0)$ and $(2, 4)$! The arc length is

$$\begin{aligned}
2 \int_0^2 \sqrt{x^2 + 1/4}\, dx &= 2 \left[\frac{1}{2} x \sqrt{x^2 + 1/4} + \frac{1}{8} \ln|2\sqrt{x^2 + 1/4} + 2x| \right]_0^2 \\
&= 2 \left(\frac{1}{2} \times 2\sqrt{4 + 1/4} + \frac{1}{8} \ln|2\sqrt{4 + 1/4} + 4| - 0 - \frac{1}{8} \ln|2\sqrt{1/4}| \right) \\
&= 2\sqrt{17/4} + \frac{1}{4} \ln(\sqrt{17} + 4) - \frac{1}{4} \ln 1 \\
&= \sqrt{17} + \frac{1}{4} \ln(\sqrt{17} + 4) \approx 4.65.
\end{aligned}$$

7. Consider the integral

$$\int_0^\pi f(x)\,dx,$$

where $f(x) = x^3 - 8x\sin x - 16\cos x$.

12 pts

(a) Let E_M be the error given by the midpoint rule with $n = 4$, when used to approximate the integral, and let E_T be the error for the trapezoidal rule with $n = 6$. Find upper bounds for $|E_M|$ and $|E_T|$.

(Your answers should be in the form $\frac{a}{b}\pi^4$ for integers a, b. You do not have to simplify the fraction further.)

Solution: We use the error bound formulae, for which we must bound $|f''\}(x)$, for $0 < x < \pi$. The derivates are

$$\begin{aligned} f'(x) &= 3x^2 - 8\sin x - 8x\cos x + 16\sin x, \\ f''(x) &= 6x - 8\cos x - 8\cos x + 8x\sin x + 16\cos x \\ &= 6x + 8x\sin x \end{aligned}$$

We now bound the second derivative.

$$\begin{aligned} |f''(x)| &= |6x + 8x\sin x| \\ &\le |6x| + |8x\sin x| && \text{(by the triangle inequality)} \\ &= 6|x| + 8|x||\sin x| \\ &\le 6\pi + 8\pi = 14\pi. \end{aligned}$$

So, our bound is $K = 14\pi$, and then the error bound formulae give

$$\begin{aligned} |E_M| &\le \frac{K(b-a)^3}{12n^2} = \frac{14\pi(\pi-0)^3}{12 \times 4^2} \\ &= \frac{14\pi^4}{12 \times 16} = \frac{14}{192}\pi^4, \\ |E_T| &\le \frac{K(b-a)^3}{24n^2} = \frac{14\pi(\pi-0)^3}{24 \times 6^2} \\ &= \frac{14\pi^4}{24 \times 36} = \frac{14}{864}\pi^4. \end{aligned}$$

8 pts

(b) Find a value of n that guarantees the midpoint and trapezoidal rule approximations of the integral are both simultaneously less than 0.25.

(You should use the approximation $\pi \approx 3$, and note that $19^2 = 361$ while $20^2 = 400$.)

Solution: Note that, in general, the bound for $|E_T|$ is smaller than (in fact, half of) the bound for $|E_M|$ for a given n. Therefore, requiring both bounds to be less than 0.25 is equivalent to requiring the (larger) $|E_M|$ bound to be less than 0.25. So, we require that

$$\frac{K(b-a)^3}{12n^2} = \frac{14\pi(\pi-0)^3}{12n^2} = \frac{14\pi^4}{12n^2} < 0.25,$$

which rearranges to

$$n > \sqrt{\frac{14\pi^4}{12 \times 0.25}} = \sqrt{\frac{14\pi^4}{3}} \approx \sqrt{\frac{14 \times 3^4}{3}} = \sqrt{14 \times 3^3} = \sqrt{14 \times 27} = \sqrt{378}.$$

Finally, observe that $19 = \sqrt{361} < \sqrt{378} < \sqrt{400} = 20$, so $n = 20$ is the smallest value of n that suffices.

8. Consider the definite integral

$$\int_1^3 \frac{3}{x^2 - x - 2}\, dx.$$

6 pts

(a) Why is this an improper integral?

Solution: The denominator factorises as $x^2 - x - 2 = (x-2)(x+1)$, and therefore the integrand is undefined at $x = -1$ and $x = 2$. In particular, this means that there is a jump discontinuity at $x = 2$, which is within the range $[1, 3]$ of integration, and thus the integral is improper.

14 pts

(b) Is this integral convergent or divergent? If it is convergent, give its value. If it is divergent, explain why this is.

Solution: We must split the integral up at $x = 2$, and take limits.

$$\int_1^3 = \lim_{a\to 2^-}\left(\int_1^a \frac{3}{(x-2)(x+1)\,dx}\right) + \lim_{b\to 2^+}\left(\int_b^3 \frac{3}{(x-2)(x+1)\,dx}\right).$$

We need to compute the indefinite integral $\int \frac{3}{(x-2)(x+1)}\,dx$ to proceed, for which we require partial fractions. Suppose that

$$\frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}.$$

Then $3 = A(x+1) + B(x-2)$. When $x = -1$, we have $3 = B(-1-2) = -3B$, so that $B = -1$. When $x = 2$, we have $3 = A(2+1) = 3A$, so that $A = 1$. Therefore,

$$\int \frac{3}{(x-2)(x+1)}\,dx = \int \frac{1}{x-2}\,dx - \int \frac{1}{x+1}\,dx$$
$$= \ln|x-2| - \ln|x+1| + c.$$

Now consider the limit in a. We have

$$\lim_{a\to 2^-}\left([\ln|x-2| - \ln|x+1|]_1^a\right) = \lim_{a\to 2^-}\left(\ln|a-2| - \ln|a+1| - \ln|1-2| + \ln|1+1|\right)$$
$$= \lim_{a\to 2^-}\left(\ln|a-2|\right) - \ln 3 + \ln 2.$$

Since the limit $\lim_{a\to 2^-}(\ln|a-2|)$ does not exist, we conclude that the integral is **divergent**.

9. Let $f(x) = \lambda(9-x)$.

10 pts

(a) Find values a, b and λ such that the function

$$g(x) = \begin{cases} f(x) & \text{when } a \le x \le b, \\ 0 & \text{otherwise.} \end{cases}$$

is a probability density function.

Solution: For $g(x)$ to be a probability density function, we require that $g(x) \ge 0$ for $a \le x \le b$, and

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_a^b g(x)\,dx = 1.$$

In order to have $g(x) \geq 0$, we need $f(x) = \lambda(9-x) \geq 0$, which is true for any $\lambda \geq 0$ when $x \leq 9$.
We can therefore choose any a, b such that $a < b \leq 9$. We will take $a = 0$ and $b = 9$.
Finally, we use the other condition to determine the value of λ:

$$\int_0^9 g(x)\,dx = \int_0^9 \lambda(9-x)\,dx = \lambda\left[9x - x^2/2\right]_0^9$$
$$= \lambda(81 - 81/2) = \frac{81}{2}\lambda = 1.$$

Therefore, we must take $\lambda = \frac{2}{81}$.

10 pts (b) For the probability distribution given by $g(x)$ (with your values of a, b and λ), find the value of the mean μ. Then write a quadratic equation in m that is satisfied by the median m.

Solution: Recall that the mean μ is given by

$$\mu = \int_{-\infty}^{\infty} xg(x)\,dx,$$

and the median m satisfies

$$\int_{-\infty}^{m} g(x) = \frac{1}{2}.$$

Then, we may calculate the mean as

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} xg(x)\,dx = \int_0^9 x\lambda(9-x)\,dx \\
&= \frac{2}{81}\left[\frac{9x^2}{2} - \frac{x^3}{3}\right]_0^9 = \frac{2}{81}\left(\frac{9^3}{2} - \frac{9^3}{3}\right) \\
&= \frac{2}{9^2}\left(9^3\left(\frac{1}{2} - \frac{1}{3}\right)\right) = 2 \times 9\left(\frac{1}{6}\right) \\
&= 3.
\end{aligned}$$

Meanwhile, m satisfies

$$\begin{aligned}
\frac{1}{2} &= \int_{-\infty} mg(x)\,dx = \int_0^m \lambda(9-x)\,dx \\
&= \frac{2}{81}\left[9x - \frac{x^2}{2}\right]_0^m = \frac{2}{81}\left(9m - \frac{m^2}{2}\right) \\
&= \frac{2}{9}m - \frac{1}{81}m^2,
\end{aligned}$$

which we may (*optionally*) rearrange to $2m^2 - 36m + 81 = 0$.