MAT 126 Final Exam

August 17, 2023

Name: ______________________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	20	20	20	20	20	40	20	20	20	200
Score:										

There are 9 problems on 12 pages in this exam (plus two cover pages). Make sure that you have them all.

Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate what is where if you expect someone to look at it. **Books, calculators, extra papers, and discussions with friends are not permitted. No electronic devices may be used AT ALL.**

Points will be taken off for writing mathematically false statements, even if the rest of the problem is correct.

Leave all answers in exact form (that is, do *not* approximate π, square roots, logarithms, and so on.) Note that you *should* know the values of the trigonometric functions at the "standard" values like $\pi/3$, as well as the values of $\ln(1)$, $\ln(e)$, e^0, $\sqrt{25}$, etc.

On the next page is a list of formulae that you are free to refer to. Maybe it will help. Or maybe not.

You have **2 hours and 30 minutes** to complete this exam.

You may freely use the following formulas.

$$\cos^2(x) + \sin^2(x) = 1$$
$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\cos^2(x) = \tfrac{1}{2}\left(1 + \cos(2x)\right)$$
$$\sin^2(x) = \tfrac{1}{2}\left(1 - \cos(2x)\right)$$
$$\tan^2(x) = \sec^2(x) - 1$$

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int u \; dv = uv - \int v du$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

$$\int \ln x \, dx = x \ln x - x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \frac{a}{a^2 + x^2} \, dx = \arctan \frac{x}{a} + C$$

$$\int \frac{a}{a^2 - x^2} \, dx = \frac{1}{2} \ln \left| \frac{x+a}{x-a} \right| + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \frac{x}{a} + C$$

$$\int \frac{a}{x\sqrt{x^2 - a^2}} \, dx = \operatorname{arcsec} \frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx \ln(x + \sqrt{x^2 - a^2}) + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln(x + \sqrt{x^2 + a^2}) + C$$

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1. Evaluate the following integrals.
   If you find an integral to be divergent, write DIVERGENT.

$\boxed{\text{10 pts}}$ (a)

$$\int x^2 e^x \, dx$$

$\boxed{\text{10 pts}}$ (b)

$$\int_1^\infty \frac{1}{t^3} \, dt$$

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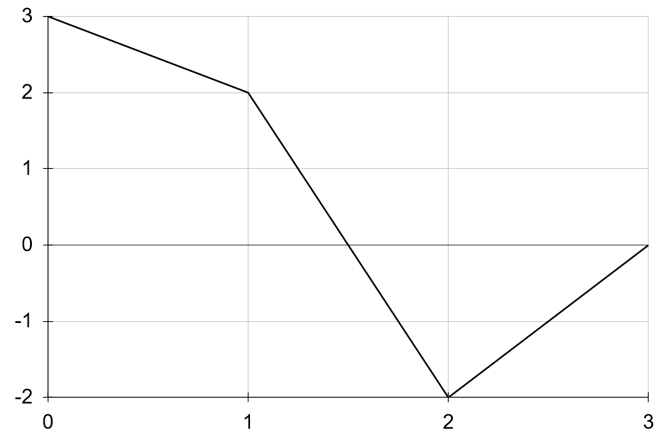
2. The graph of $y = f(x)$ is shown below.



Let $g(x) = \int_0^x f(t)\,dt$. What are the values of the following?

6 pts (a) $g(1)$

6 pts (b) $g(3)$

8 pts (c) $g'(2)$

20 pts 3. Consider the region enclosed by the lines $y = 3x, y = 4x$ and $x = 3$. What is the volume of the solid obtained by rotating this region about the x-axis?

$\boxed{\text{20 pts}}$ 4. Let $f(x)$ be an increasing function on the interval $[0, 1]$. Rank the following in order, from smallest value to largest value:

i) $f(0)$

ii) $f(1)$

iii) The approximation of $\int_0^1 f(x)\,dx$ using left end-points and $n = 5$.

iv) The approximation of $\int_0^1 f(x)\,dx$ using right end-points and $n = 5$.

v) The average value of $f(x)$ on $[0, 1]$.

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5. Consider the curve $y = x^2$.

10 pts (a) Find functions $f(t), g(t)$ such that the curve is described parametrically by

$$x = f(t),$$
$$y = g(t).$$

10 pts (b) Write an integral that gives the length of this curve between the points $(0, 0)$ and $(2, 4)$. [You do not have to evaluate this integral.]

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6. Consider the integral $\int \sqrt{x^2 + \frac{1}{4}}\, dx$.

6 pts

(a) Using the substitution $x = \frac{1}{2}\tan\theta$, show that

$$\int \sqrt{x^2 + \frac{1}{4}}\, dx = \frac{1}{4}\int \sec^3\theta\, d\theta.$$

10 pts

(b) Using integration by parts, show that

$$2\int \sec^3\theta\, d\theta = \tan\theta\sec\theta + \int \sec\theta\, d\theta.$$

[Hint: Consider $\int \sec^3\theta\, d\theta$ for the integration by parts, and recall that $\frac{d}{d\theta}(\sec\theta) = \tan\theta\sec\theta$.]

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6 pts (c) Find $\frac{d}{d\theta}\left(\sec\theta + \tan\theta\right)$.

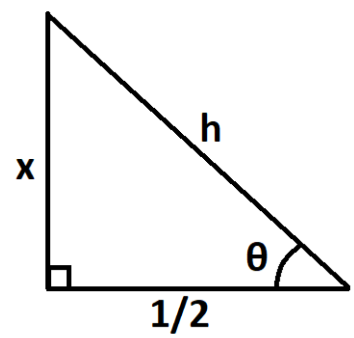
10 pts (d) By writing $\sec\theta$ as

$$\sec\theta = \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta},$$

show that

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + c.$$

4 pts (e) Consider the below triangle.



Find the values of $h, \tan\theta, \cos\theta$, and $\sec\theta$ (in terms of x).

[Do NOT physically measure the triangle; it is not intended to be drawn to scale.]

4 pts (f) Evaluate the integral $\int \sqrt{x^2 + \frac{1}{4}}\,dx$.
(i.e. find a function $F(x)$ such that $F'(x) = \sqrt{x^2 + \frac{1}{4}}$.)

7. Consider the integral

$$\int_0^{\pi} f(x)\,dx,$$

where $f(x) = x^3 - 8x\sin x - 16\cos x$.

12 pts

(a) Let E_M be the error given by the midpoint rule with $n = 4$, when used to approximate the integral, and let E_T be the error for the trapezoidal rule with $n = 6$. Find upper bounds for $|E_M|$ and $|E_T|$.

(Your answers should be in the form $\frac{a}{b}\pi^4$ for integers a, b. You do not have to simplify the fraction further.)

8 pts

(b) Find a value of n that guarantees the midpoint and trapezoidal rule approximations of the integral are both simultaneously less than 0.25.

(You should use the approximation $\pi \approx 3$, and note that $19^2 = 361$ while $20^2 = 400$.)

8. Consider the definite integral

$$\int_1^3 \frac{3}{x^2 - x - 2}\, dx.$$

6 pts

(a) Why is this an improper integral?

14 pts

(b) Is this integral convergent or divergent? If it is convergent, give its value. If it is divergent, explain why this is.

9. Let $f(x) = \lambda(9 - x)$.

10 pts (a) Find values a, b and λ such that the function

$$g(x) = \begin{cases} f(x) & \text{when } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

is a probability density function.

10 pts (b) For the probability distribution given by $g(x)$ (with your values of a, b and λ), find the value of the mean μ. Then write a quadratic equation in m that is satisfied by the median m.