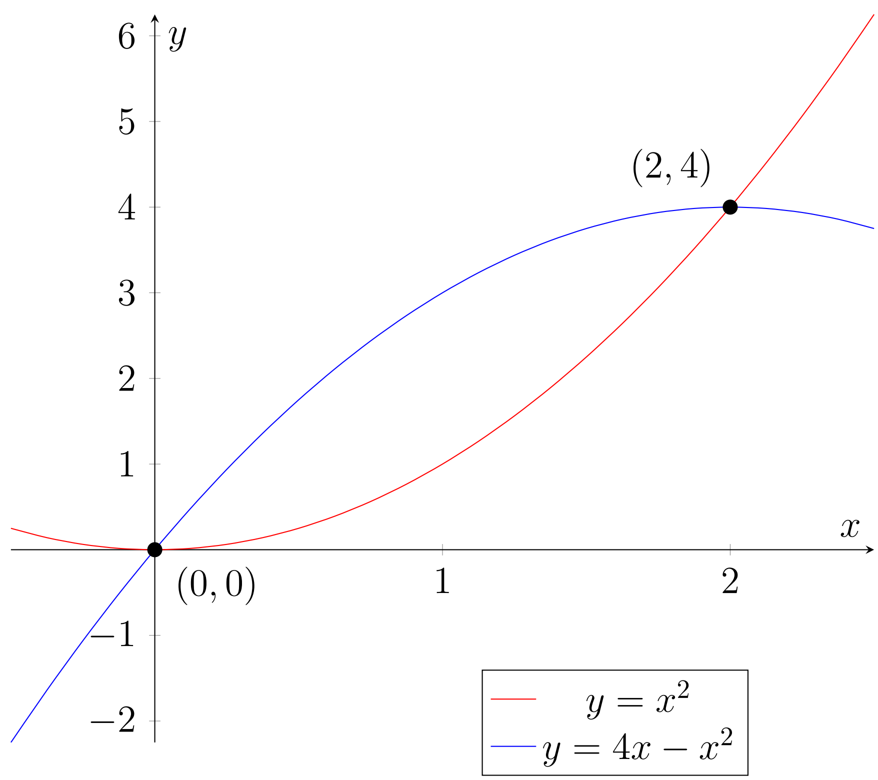
MAT126 Homework 4 Solutions

Problem 1. Find the area of the bounded region lying between the curves $y = x^2$ and $y = 4x - x^2$.

Solution: The curves intersect when $x^2 = 4x - x^2$, which we solve to get $x = 0$ and $x = 2$. Plugging these values into either of our curve equations gives $y = 0$ and $y = 4$, respectively. With this information, and knowing the shapes of the curves from their equations, we may sketch the situation.

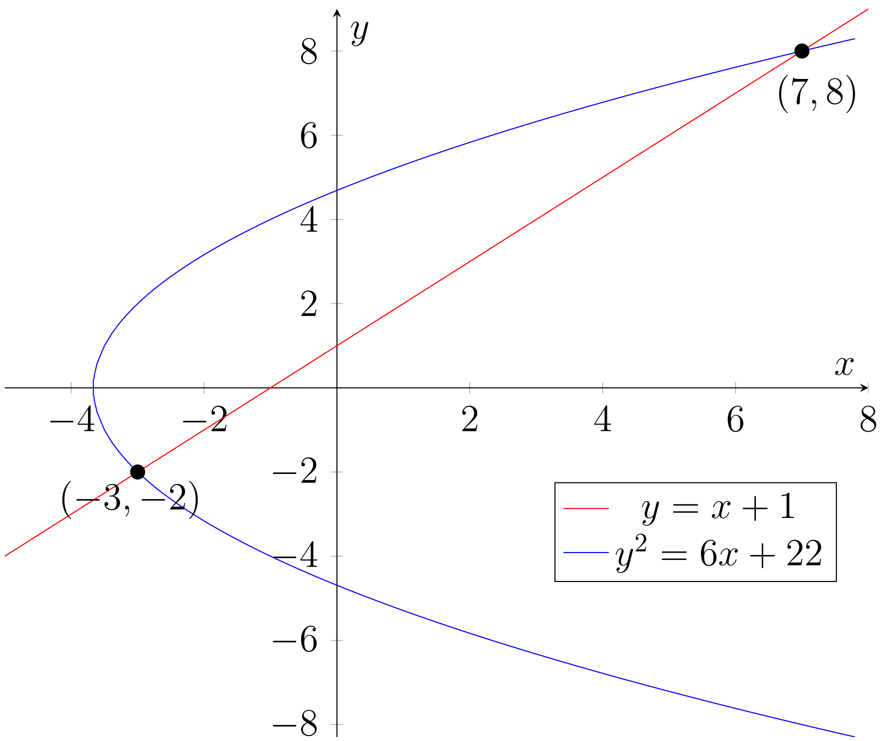


The area of the region is then given by

$$\begin{aligned} A &= \int_0^2 4x - x^2 - x^2 \, dx \\ &= \left[2x^2 - \frac{2}{3}x^3 \right]_0^2 \\ &= 8/3. \end{aligned}$$

Problem 2. Find the area of the region enclosed by the line $y = x + 1$ and the parabola $y^2 = 6x + 22$.

Solution: Again, we find the points of intersection and sketch the region in question. The line and parabola intersect when $(x + 1)^2 = 6x + 22$, which solves to $x = -3$ and $x = 7$. The points of intersection are $(-3, -2)$ and $(7, 8)$. The region looks something like this:



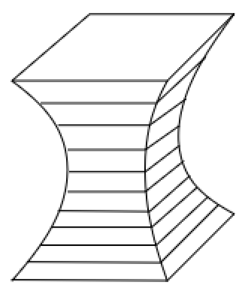
If we were to integrate with respect to x, the curve at the bottom of the region would change midway through. This is perfectly doable, but we proceed with the cleaner method of integrating with respect to y. First, we write our curve equations as functions of y, which take the form $x = y - 1$ and $x = (y^2 - 22)/6$.

The area of the region is then given by

$$\begin{aligned}
A &= \int_{-2}^{8} y - 1 - y^2/6 + 22/6\, dy \\
&= \left[\frac{y^2}{2} - y - \frac{y^3}{18} + \frac{22y}{6}\right]_{-2}^{8} \\
&= 250/9.
\end{aligned}$$

Problem 3. A pillar that is π feet tall is made so that every horizontal cross-section at height h is a square of side length

$$s(h) = 3 - \sin(h).$$



(a) Write an integral which represents the volume of the pillar.

(b) Evaluate the integral to find the volume of the pillar.

Solution: (a) The volume V is given by $V = \int_0^\pi A(h)\, dh$, where $A(h)$ is the cross-sectional area at height h. Since cross-sections are squares with side length $s(h)$, we find that

$$\begin{aligned}
V &= \int_0^\pi (s(h))^2\, dh \\
&= \int_0^\pi 9 - 6\sin(h) + \sin^2(h)\, dh.
\end{aligned}$$

(b)

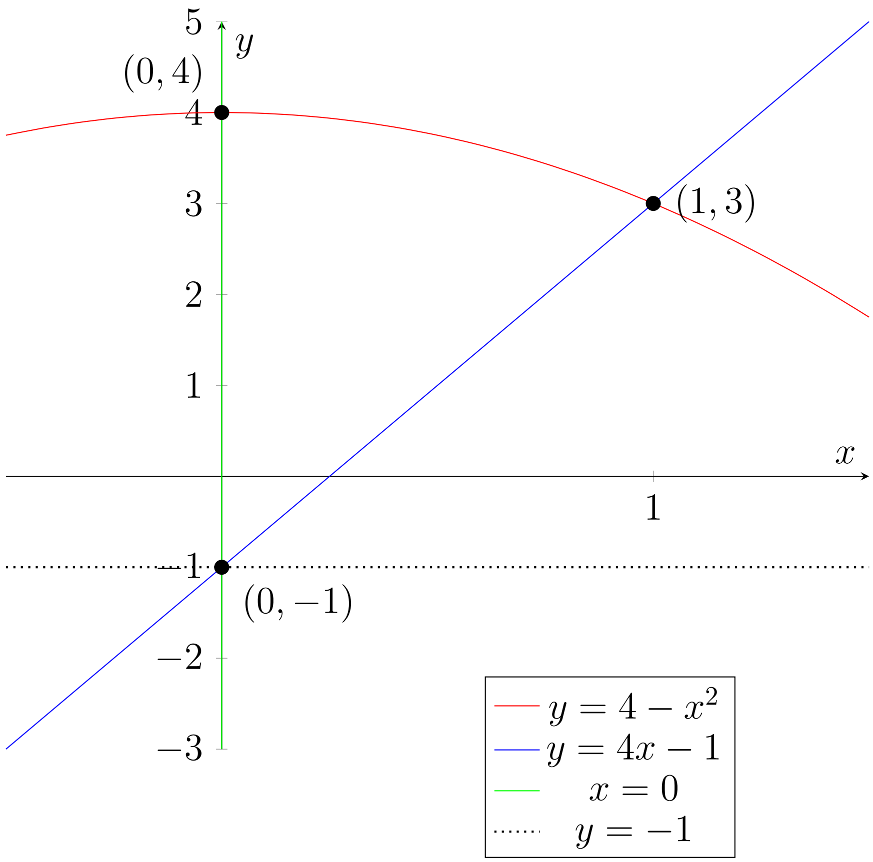
$$
\begin{aligned}
V &= \int_0^{\pi} 9 - 6\sin(h) + \sin^2(h)\,dh \\
&= \int_0^{\pi} 9 - 6\sin(h) + \frac{1}{2}\left(1 - \cos(2h)\right)\,dh \\
&= \int_0^{\pi} \frac{19}{2} - 6\sin(h) - \frac{1}{2}\cos(2h)\,dh \\
&= \left[\frac{19}{2}h + 6\cos(h) - \frac{1}{4}\sin(2h)\right]_0^{\pi} \\
&= \frac{19}{2}\pi - 6 - 6 \\
&= \frac{19}{2}\pi - 12.
\end{aligned}
$$

Problem 4. Consider the region bounded by the below curves, with $x > 0$.

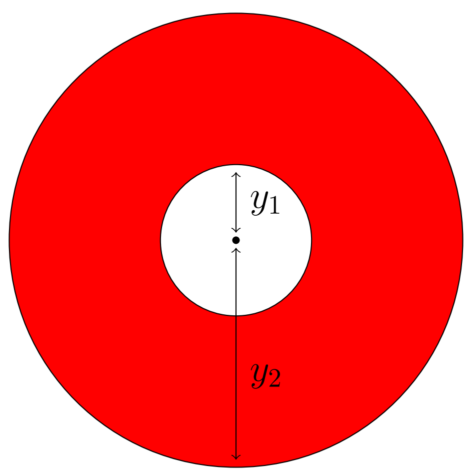
$$y = 4 - x^2, \quad y = 4x - 1, \quad x = 0.$$

What is the volume of the solid obtained by rotating this region about the line $y = -1$?

Solution: Since $x > 0$, we need only graph the situation to the right of $x = 0$ (which, we note, is the y-axis). The other two curves intersect when $4 - x^2 = 4x - 1$, which we solve to get $x = -5$ and $x = 1$. Since the region only involves x values bigger than 0, we care only about $x = 1$, which has $y = 3$ on both of the curves $y = 4 - x^2$ and $y = 4x - 1$. We are ready to sketch the region.



After rotating the region about $y = -1$, a typical cross-section cut parallel to the y-axis (say, at $x = x_0$) looks like this:



where $y_1 = 4x_0$ is the distance from $(x_0, -1)$ to $(x_0, 4x_0 - 1)$, and $y_2 = 5 - x_0^2$ is the distance from $(x_0, -1)$ to $(x_0, 4 - x_0^2)$. It follows that, in general, the area of an arbitrary cross-section is given by

$$\begin{aligned} A(x) &= \pi(y_2^2 - y_1^2) \\ &= \pi(25 - 10x^2 + x^4 - 16x^2) \\ &= \pi(x^4 - 26x^2 + 25). \end{aligned}$$

Finally, we integrate over the x-values in the region to get the solid's volume.

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &= \pi \int_0^1 x^4 - 26x^2 + 25\,dx \\ &= \pi \left[\frac{x^5}{5} - \frac{26x^3}{3} + 25x \right]_0^1 \\ &= \pi \left(\frac{1}{5} - \frac{26}{3} + 25 \right) = \frac{248}{15}\pi. \end{aligned}$$

Problem 5. Let R be the region between $y = x^2$ and $y = 2x$.

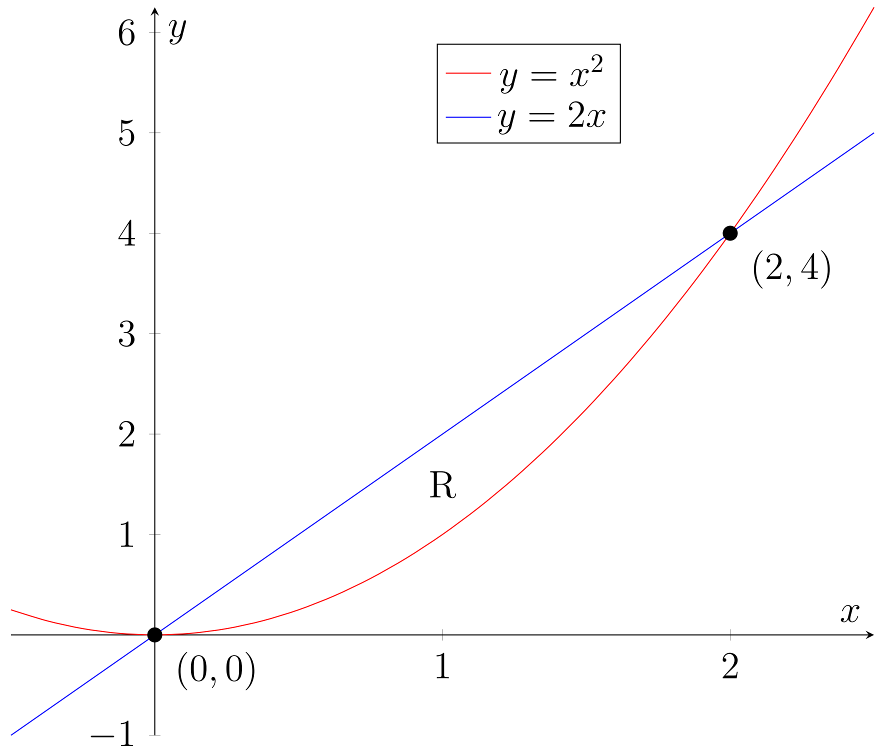
(a) Find the points of intersection of these two curves. (Okay, one of them is a line; lines are just straight curves!!)

(b) Sketch the region R. Pick appropriate scales for x and y.

(c) Use cylindrical shells to find the volume of the solid generated when the region R is revolved about the x-axis.

Solution: (a) We have $x^2 = 2x$, so $x = 0$ and $x = 2$. Thus the points of intersection are $(0, 0)$ and $(2, 4)$.

(b) We sketch R below.



(c) To use shells, we must write the equations of the curves as functions of y. Noting that the boundary of our region always has non-negative x and y values (so that we must take the positive square root), we get $x = \sqrt{y}$ and $x = y/2$. Finally, then, we find that

$$\begin{aligned} V = &= \pi \int_0^4 (\sqrt{y})^2 - \left(\frac{1}{2}y\right)^2 \, dy \\ &= \pi \int_0^4 y - \frac{1}{4}y^2 \, dy \\ &= \pi \left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 \\ &= \pi \left(\frac{16}{2} - \frac{64}{12}\right) \\ &= \frac{8}{3}\pi. \end{aligned}$$