MAT126 Homework 3 Solutions

Problem 1. After performing a long division, evaluate the integral

$$\int \frac{x^5 + 4x^3 + 2x^2 + 1}{x^2 + 1}\, dx.$$

Solution: Long division gives

$$\frac{x^5 + 4x^3 + 2x^2 + 1}{x^2 + 1} = x^3 + 3x + 2 - \frac{3x + 1}{x^2 + 1},$$

and therefore

$$\begin{aligned}\int \frac{x^5 + 4x^3 + 2x^2 + 1}{x^2 + 1}\, dx &= x^4/4 + 3x^2/2 + 2x - \int \frac{3x + 1}{x^2 + 1}\, dx \\ &= x^4/4 + 3x^2/2 + 2x \\ &\quad - \frac{3}{2}\int \frac{2x}{x^2 + 1}\, dx - \int \frac{1}{x^2 + 1}\, dx.\end{aligned}$$

Of these two integral we have left to evaluate, we have factored 3/2 out of the first one so that the derivative of $x^2 + 1$ is on the numerator, and thus the integral will be $\ln|x^2 + 1|$. For the other integral, recall that when working with partial fractions, we will frequently encounter the integral

$$\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c.$$

(This will be given to you in the final.) In our case, $a = 1$ gives us our answer. In summary, we have

$$\int \frac{x^5 + 4x^3 + 2x^2 + 1}{x^2 + 1}\, dx = \frac{x^4}{4} + \frac{3x^2}{2} + 2x - \frac{3}{2}\ln|x^2 + 1| + \tan^{-1}(x) + c.$$

Problem 2. Consider the integral

$$\int \frac{x + 2}{x^3 - 6x^2 + 9x}\, dx.$$

(a) Write the integrand in its partial fraction form, and find the resulting constants A, B and C.

(b) Evaluate the integral.

Solution: (a) The denominator factorizes as $x^3 - 6x^2 + 9x = x(x-3)^2$, so we have the repeated linear factor $(x-3)$. The corresponding partial fraction form is

$$\frac{x+2}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}.$$

Multiplying through $x(x-3)^2$ gives

$$x + 2 = A(x-3)^2 + Bx(x-3) + Cx.$$

When $x = 0$, this gives $2 = 9A$, so $A = 2/9$. When $x = 3$, we instead get $5 = 3C$, so $C = 5/3$. To find B, we could set x to some other value, e.g. $x = 1$, and use the values of A and C. In this case, it is quicker to 'equate coefficients'. In the equation above, the LHS and RHS are identically equal, so the coefficients of x^2, x and 1 must be the same on each side. The coefficient of x^2 is 0 on the left, and $A + B$ on the right, so $A + B = 0$ and thus $B = -A = -2/9$. Therefore,

$$\frac{x+2}{x(x-3)^2} = \frac{2/9}{x} + \frac{-2/9}{x-3} + \frac{5/3}{(x-3)^2}.$$

(b) We have three integrals to evaluate. The first two are hopefully familiar by now. For the third one, we use the substitution $u = x - 3$.

$$\begin{aligned}
\int \frac{x+2}{x^3 - 6x^2 + 9x}\,dx &= \frac{2/9}{\int}\frac{1}{x}\,dx - \frac{2/9}{\int}\frac{1}{x-3}\,dx + \frac{5/3}{\int}\frac{1}{(x-3)^2}\,dx \\
&= \frac{2}{9}\ln|x| - \frac{2}{9}\ln|x-3| + \frac{5}{3}\int \frac{1}{u^2}\,du \\
&= \frac{2}{9}\ln|x| - \frac{2}{9}\ln|x-3| + \frac{5}{3}\cdot\frac{-1}{2}u^{-3} + c \\
&= \frac{2}{9}\ln|x| - \frac{2}{9}\ln|x-3| - \frac{5}{6}\cdot\frac{1}{(x-3)^3} + c.
\end{aligned}$$

Problem 3. Consider the integral

$$\int_0^{3/2} x\cos(x^2)\,dx.$$

(a) Approximate the integral using the mid-point rule with $n = 5$. Give your answer to 4 decimal places.

(b) Approximate the integral using the trapezoidal rule with $n = 15$. Give your answer to 4 decimal places.

(c) Denote the error in your approximations by E_M and E_T, respectively. By first finding a bound for the second derivative of $f(x) = x\cos(x^2)$, use the error bounds we have seen in class to bound $|E_M|$ and $|E_T|$. Leave your answers in exact form.

(d) What value of n should we choose to guarantee that both the mid-point rule and trapezoidal rule will have error less than 0.01?

Solution: (a) Since we are using the mid-point rule, we must first find the mid-points. With $n = 5$, there will be 5 sub-intervals and thus 5 mid-points; call them x_1, x_2, x_3 and x_5. We are integrating over the interval $[0, 3/2]$, so $\Delta x = (b - a)/n = (3/2 - 0)/5 = 3/10$. Thus, the first sub-interval is $[0, 3/10]$. We can find the mid-point of this sub-interval by averaging the two end-points, giving $x_1 = (0 + 3/10)/2 = 3/20$. Then, since each mid-point will be a distance of Δx apart, we get $x_2 = x_1 + \Delta x = 9/20, x_3 = x_2 + \Delta x = 15/20 = 3/4, x_4 = x_3 + \Delta x = 21/20$ and finally $x_5 = x_4 + \Delta x = 27/20$. All that remains is to plug these values into the mid-point rule formula.

Letting $f(x) = x\cos(x^2)$, we get

$$\begin{aligned}\int_0^{3/2} x\cos(x^2)\,dx &\approx \Delta x(f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)) \\ &= \frac{3}{10}\left(\frac{3}{20}\cos\left(\left(\frac{3}{20}\right)^2\right)+\frac{9}{20}\cos\left(\left(\frac{9}{20}\right)^2\right)\right. \\ &\quad +\frac{3}{4}\cos\left(\left(\frac{3}{4}\right)^2\right)+\frac{21}{20}\cos\left(\left(\frac{21}{20}\right)^2\right) \\ &\quad \left.+\frac{27}{20}\cos\left(\left(\frac{27}{20}\right)^2\right)\right).\end{aligned}$$

Plugging this into our calculator gives our approximation, to 4 decimal places as requested, as

$$\int_0^{3/2} x\cos(x^2)\,dx \approx 0.4089.$$

(b) For the trapezoidal rule, we need all 16 of the end-points from our 15 sub-intervals. Firstly, we have $\Delta x = (b-a)/n = (3/2-0)/15 = 1/10$. Calling the end-points $x_0, x_1, \ldots, x_15$, we see that $x_i = i\Delta x = i/10$ for $i = 0, 1, \ldots, 15$. The formula for the trapezoidal rule is

$$\begin{aligned}\int_0^{3/2} x\cos(x^2)\,dx \approx \frac{\Delta x}{2}(f(x_0)+2f(x_1)+2f(x_2)+\ldots \\ +2f(x_{14})+f(x_{15}))\end{aligned}$$

Now, one could write this out in full and enter it into a calculator. It is quicker, though, to use some online summation calculator, for instance WolframAlpha. Typing in "sum of $\frac{i}{10}\cos((\frac{i}{10})^2)$ from $i = 1$ to $i = 14$" gives 4.31866584...

Now, since the terms $f(x_1), f(x_2), \ldots, f(x_{14})$ appear twice, we multiply this by 2. Then we still have $f(x_0)$ and $f(x_{15})$ to add on. We find that

$$f(x_0) = f(0) = 0\cos(0^2) = 0,$$
$$f(x_{15}) = f(3/2) = \frac{3}{2}\cos\left(\frac{9}{4}\right),$$

so adding this on to 2×4.31866584 gives 7.69507125...

Finally, we multiply by $\Delta x/2$ to get

$$\int_0^{3/2} x\cos(x^2)\,dx \approx 0.3848.$$

(c) Recall that our error bound method requires us to first bound the second derivative of $f(x) = x\cos(x^2)$. So, first we must differentiate. The product and chain rules give

$$f'(x) = \cos(x^2) - 2x^2\sin(x^2),$$

and then

$$\begin{aligned} f''(x) &= -2x\sin(x^2) - (4x\sin(x^2) + 4x^3\cos(x^2)) \\ &= -6x\sin(x^2) - 4x^3\cos(x^2). \end{aligned}$$

Now our job is to find K (preferably as small as possible) such that $|f''(x)| < K$. First, we use the rules $|a+b| \leq |a| + |b|$ and $|ab| = |a||b|$ to get

$$\begin{aligned} |f''(x)| &= |-6x\sin(x^2) - 4x^3\cos(x^2)| \\ &\leq |-6x\sin(x^2)| + |-4x^3\cos(x^2)| \\ &= |-6||x||\sin(x^2)| + |-4||x^3||\cos(x^2)| \\ &= 6|x||\sin(x^2)| + 4|x^3||\cos(x^2)|. \end{aligned}$$

In our integral, we have $0 \leq x \leq 3/2$. For this range of x-values, the largest value that $|x|$ attains is $3/2$. The largest value that $|x^3|$ attains is $(3/2)^3 = 27/8$. A common mistake is to then assume that the largest values are always attained when x is at its largest, but this is not the case for $|\sin(x^2)|$ and $|\cos(x^2)|$.

Both of these functions range between 0 and 1, so their largest value is 1 (which is not at $x = 3/2$ ...). Putting all these largest values in, we find

$$|f''(x)| \leq 6 \cdot \frac{3}{2} \cdot 1 + 4 \cdot \frac{27}{8} \cdot 1 = 45/2.$$

Thus, we take $K = 45/2$, and using our error bound formulae, we get

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{\frac{45}{2} \cdot (3/2)^3}{24 \cdot 5^2} = \frac{81}{640},$$

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{\frac{45}{2} \cdot (3/2)^3}{12 \cdot 15^2} = 9/320.$$

(d) First note that, for a given n, our bound on the error for the midpoint rule is half our bound on the error for the trapezoidal rule. Thus, if we want the error to be less than 0.01 for both simultaneously, we can just make the error bound for the trapezoidal rule less than 0.01, and then the error bound for the midpoint rule will be less than 0.005 (and so certainly less than 0.01). So, we require that

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} < 0.01.$$

Now we just need to plug in our values for K, a and b, and rearrange the inequality for n. So,

$$\frac{1215/16}{12n^2} < 0.01,$$

which rearranges to

$$n > \sqrt{\frac{1215/16}{0.12}} = 25.1558...$$

Therefore, the smallest integer n that we can choose so that our errors are less than 0.01 is $n = 26$.

Problem 4. For each of the following integrals, explain why they are improper integrals. Then, evaluate them. If they are divergent, then write 'DIVERGENT'.

(a)

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

(b)

$$\int_2^4 \frac{2x}{x^2-9}\,dx.$$

(c)

$$\int_0^\infty (1-x)e^{-x}\,dx$$

Hint: Proceed with integration by parts, and make use of the limit

$$\lim_{t\to\infty} te^{-t} = 0.$$

(d)

$$\int_0^{\pi/2} \tan\theta\,d\theta.$$

Hint: Write $\tan\theta$ as a fraction involving $\sin\theta$ and $\cos\theta$, and use a substitution.

Solution: (a) This integral is improper, since $1/\sqrt{x}\to\infty$ as $x\to 0$. In other words, the integrand has an infinite discontinuity at $x=0$, which is within the range of integration. To get around this, we re-write the integral as a limit in the following way:

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0}\left(\int_t^1 \frac{1}{\sqrt{x}}\,dx\right) &= \lim_{t\to 0}\left(\left[2x^{1/2}\right]_t^1\right)\\ &= \lim_{t\to 0}\left(2-2\sqrt{t}\right) = 2.\end{aligned}$$

(b) This integral is improper because the integrand has an infinite discontinuity at $x=3$, which is within the range of integration. To see that this infinite discontinuity is there, note that the denominator factorizes as $x^2 - 9 = (x-3)(x+3)$, so is zero at $x = -3$ and $x = 3$.

This time, since the infinite discontinuity is in the middle of the range of integration, we must split the integral up here and then use limits:

$$\begin{aligned} \int_2^4 \frac{2x}{x^2-9}\,dx &= \int_2^3 \frac{2x}{x^2-9}\,dx + \int_3^4 \frac{2x}{x^2-9}\,dx \\ &= \lim_{s\to 3}\left(\int_2^s \frac{2x}{x^2-9}\,dx\right) + \lim_{t\to 3}\left(\int_t^4 \frac{2x}{x^2-9}\,dx\right). \end{aligned}$$

Let's just work with the second limit / integral for now; if that limit doesn't exist, then we can immediately deduce the integral is divergent and be finished. If the limit does exist, then we will have to go on and check the first limit / integral as well. We have the integral inside the limit evaluating to

$$\begin{aligned} \int_t^4 \frac{2x}{x^2-9}\,dx &= \left[\ln|x^2-9|\right]_t^4 \\ &= \ln|16-9| - \ln|t^2-9|. \end{aligned}$$

In the limit, $t \to 3$, and when this happens, $\ln|t^2-9| \to -\infty$, so the limit doesn't exist! Therefore, the integral is **divergent**.

(c) The integral is improper, since it has an infinite range of integration (i.e. infinity appears as one of the limits of the integral). Now to evaluate the integral!
Let's first solve the corresponding indefinite integral using integration by parts. Set $u = 1 - x$ and $v' = e^{-x}$. Then $u' = -1$

and $v = -e^{-x}$, so

$$\begin{aligned}\int (1-x)e^{-x}\,dx &= (1-x)(-e^{-x}) - \int (-1)(-e^{-x})\,dx \\ &= (x-1)(e^{-x}) - \int e^{-x}\,dx \\ &= xe^{-x} - e^{-x} - (-e^{-x}) + c \\ &= xe^{-x} - e^{-x} + e^{-x} + c \\ &= xe^{-x} + c.\end{aligned}$$

Now for the definite integral. We have

$$\begin{aligned}\int_0^\infty (1-x)e^{-x}\,dx &= \lim_{t\to\infty}\left(\int_0^t (1-x)e^{-x}\,dx\right) \\ &= \lim_{t\to\infty}\left(\left[xe^{-x}\right]_0^t\right) \\ &= \lim_{t\to\infty}\left(te^{-t}\right) \\ &= 0.\end{aligned}$$

(d) The integral is improper since $\tan\theta$ has an infinite discontinuity at $\theta = \pi/2$, which is within the range of integration. We have $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Note that the numerator here is -1 times the derivative of the denominator, so the integral will be $-\ln|\cos\theta| + c$. (If you prefer, use the substitution $u = \cos\theta$.) Thus, we have

$$\begin{aligned}\int_0^{\pi/2} \tan\theta\,d\theta &= \lim_{t\to\pi/2}\left(\int_0^t \tan\theta\,d\theta\right) \\ &= \lim_{t\to\pi/2}\left([-\ln|\cos\theta|]_0^t\right) \\ &= \lim_{t\to\pi/2}\left(-\ln|\cos t| + \ln|\cos 0|\right) \\ &= \lim_{t\to\pi/2}\left(-\ln|\cos t| + \ln|1|\right) \\ &= \lim_{t\to\pi/2}\left(-\ln|\cos t|\right).\end{aligned}$$

As $t \to \pi/2$, $\cos t \to 0$, so that $\ln|\cos t| \to -\infty$. Therefore, the limit does not exist, and the integral is **divergent**.