MAT126 Homework 2 Solutions

Problem 1. Evaluate the following integrals:

(a)

$$\int_3^5 \frac{1}{x+3}\,dx.$$

(b)

$$\int e^x \sin(x)\,dx.$$

(c)

$$\int \tan^4 x \sec^6 x\,dx.$$

(d)

$$\int \frac{8x-1}{5x^2+2x-3}.$$

Solution: (a) We use the substitution $u = x+3$. Note that we have to substitute out the upper and lower limits as well. So, $du = dx$, and when $x = 3$, we have $u = 3+3 = 6$, while when $x = 5$, we have $u = 5+3 = 8$. Therefore,

$$\begin{aligned}\int_3^5 \frac{1}{x+3}\,dx = \int_6^8 \frac{1}{u}\,du = \Big[\ln|u|\Big]_6^8 &= \ln|8| - \ln|6| = \ln 8 - \ln 6 \\ &= \ln(8/6) = \ln(4/3).\end{aligned}$$

(b) We use integration by parts, setting $u = \sin(x)$ and $v' = e^x$. Then we get $u' = \cos(x)$ and $v = e^x$, so

$$\int e^x \sin(x)\,dx = uv - \int u'v\,dx = e^x \sin(x) - \int e^x \cos(x)\,dx.$$

The integral on the RHS is not any easier to solve than the one we started with; we need to use integration by parts again, on this integral. Setting $u = \cos(x)$ and $v' = e^x$ gives $u' = -\sin(x)$ and $v = e^x$, so

$$\begin{aligned}\int e^x \cos(x)\,dx &= e^x \cos(x) - \int e^x(-\sin(x))\,dx \\ &= e^x \cos(x) + \int e^x \sin(x)\,dx.\end{aligned}$$

If we denote our integral by $I = \int e^x \sin(x)\,dx$, then we have shown that $I = e^x \sin(x) - e^x \cos(x) - I$, so we can rearrange for I to solve the integral:

$$\int e^x \sin(x)\,dx = I = \frac{1}{2}\Big(e^x \sin(x) - e^x \cos(x)\Big) + c.$$

(c) We try the substitution $u = \tan(x)$, giving $dx = du/\sec^2(x)$. Also using the identity $\tan^2(x) + 1 = \sec^2(x)$, we get

$$\begin{aligned}\int \tan^4(x)\sec^6(x)\,dx &= \int u^4 \sec^6(x) \cdot \frac{1}{\sec^2(x)}\,du \\ &= \int u^4 \sec^4(x)\,du = \int u^4(\tan^2(x)+1)^2\,du \\ &= \int u^4(u^2+1)^2\,du = \int u^4(u^4 + 2u^2 + 1)\,du \\ &= \int u^8 + 2u^6 + u^4\,du = \frac{1}{9}u^9 + \frac{2}{7}u^7 + \frac{1}{5}u^5 + c \\ &= \frac{1}{9}\tan^9(x) + \frac{2}{7}\tan^7(x) + \frac{1}{5}\tan^5(x) + c.\end{aligned}$$

(d) The denominator factorizes as $5x^2+2x-3 = (5x-3)(x+1)$, so we proceed with the below partial fraction.

$$\frac{8x-1}{5x^2+2x-3} = \frac{A}{5x-3} + \frac{B}{x+1}.$$

Multiplying through by the denominator then gives $8x - 1 = A(x+1) + B(5x-3)$. This must hold for all x, so to find A and B we can choose any values of x we like and then solve for these constants. When $x = -1$, we have $-9 = -8B$, so $B = 9/8$.

When $x = 0$, we have $-1 = A - 3B$, so $A = 3B - 1$. Using the value of B that we have already found, we get $A = 27/8 - 1 = 19/8$. Therefore, we have

$$\int \frac{8x-1}{5x^2+2x-3}\,dx = \frac{19}{8}\int \frac{1}{5x-3}\,dx + \frac{9}{8}\int \frac{1}{x+1}\,dx.$$

We solve these with the substitutions $u = 5x - 3$ and $v = x + 1$ respectively, to give

$$\int \frac{8x-1}{5x^2+2x-3}\,dx = \frac{19}{40}\ln|5x-3| + \frac{9}{8}\ln|x+1| + c.$$

Problem 2. Let $f(x)$ be any differentiable function. Prove the below formula, using a substitution.

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.$$

Solution: We use the substitution $u = f(x)$. Then $dx = du/f'(x)$, so

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{f'(x)}{u}\cdot\frac{1}{f'(x)}\,du = \int \frac{1}{u}\,du = \ln|u|+c = \ln|f(x)|+c.$$

Problem 3. Consider the integral

$$\int \frac{x+6}{4x^3-7x^2-15x}\,dx.$$

Solve it by writing the integrand (this means the function that is being integrated) in the form

$$\frac{A}{x} + \frac{B}{x-3} + \frac{C}{4x+5}.$$

Solution: Note the denominator factorizes as $4x^3-7x^2-15x = x(4x^2-7x-15) = x(x-3)(4x+5)$. Writing the integrand in the suggested form and multiplying through by the denominator gives

$$x+6 = A(x-3)(4x+5) + Bx(4x+5) + Cx(x-3).$$

When $x=0$, this gives $6 = A(-3)(5) = -15A$, so $A = -6/15 = -2/5$. When $x=3$, we instead get $9 = B(3)(17) = 51B$, so $B = 9/51 = 3/17$. Finally, we can set x to be anything to find C; let's do $x=1$. Then $7 = A(-2)(9) + B(1)(9) + C(1)(-2) = -18A + 9B - 2C$. Therefore,

$$2C = -18 \cdot \frac{-2}{5} + 9 \cdot \frac{3}{17} - 7 = \frac{152}{85},$$

and thus $C = 76/85$. Therefore,

$$\int \frac{x+6}{4x^3-7x^2-15x}\,dx = -\frac{2}{5}\int \frac{1}{x}\,dx + \frac{3}{17}\int \frac{1}{x-3}\,dx + \frac{19}{85}\ln|4x+5| + c,$$

where we have used the substitutions $u = x-3$ and $v = 4x+5$ to solve the latter two integrals. Note that the v-substitution adds a factor of $1/4$ to the final answer.

Problem 4. Consider the integral

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx.$$

(a) According to lectures, which substitution should we try if we see the term $\sqrt{x^2+a^2}$?

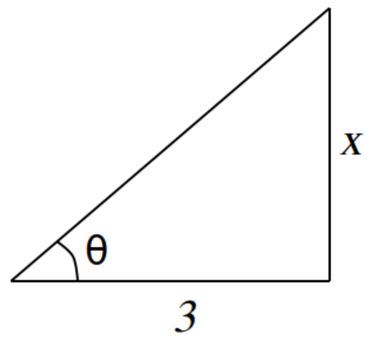
(b) Make a substitution to show that our integral is equal to

$$27\int \tan^3\theta \sec\theta\,d\theta.$$

(c) Using another substitution, show that

$$27\int \tan^3\theta \sec\theta\,d\theta = 9\sec^3\theta - 27\sec\theta + c.$$

(d) Suppose we have the following triangle:



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Using SOH CAH TOA, what is the value of $\tan\theta$? What is the value of the hypotenuse? What is the value of $\cos\theta$? What is the value of $\sec\theta$?

(e) Evaluate the integral given at the start of the question.

Solution: (a) In such a situation, we should try the substitution $x = a\tan(\theta)$.

(b) We try the substitution suggested in part a, noting for this integral we have $a = 3$. So $x = 3\tan(\theta)$, and $dx = 3\sec^2(\theta)$.

Recalling the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$, this gives

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^2+9}}\,dx &= \int \frac{(3\tan(\theta))^3}{\sqrt{(3\tan(\theta))^2+9}} \cdot 3\sec^2(\theta)\,d\theta \\
&= 3\int \frac{27\tan^3(\theta)}{\sqrt{9\tan^2(\theta)+9}}\sec^2(\theta)\,d\theta \\
&= 3\cdot 27\int \frac{\tan^3(\theta)}{\sqrt{9(\tan^2(\theta)+1)}}\sec^2(\theta)\,d\theta \\
&= 3\cdot 27\int \frac{\tan^3(\theta)}{3\sqrt{\tan^2(\theta)+1}}\sec^2(\theta)\,d\theta \\
&= 27\int \frac{\tan^3(\theta)}{\sqrt{\sec^2(\theta)}}\sec^2(\theta)\,d\theta \\
&= 27\int \frac{\tan^3(\theta)}{\sec(\theta)}\sec^2(\theta)\,d\theta \\
&= 27\int \tan^3(\theta)\sec(\theta)\,d\theta.
\end{aligned}$$

(c) Now we use the substitution $u = \sec(\theta)$. Then $\frac{du}{d\theta} = \tan(\theta)\sec(\theta)$, so $d\theta = du/\tan(\theta)\sec(\theta)$, so we find that

$$\begin{aligned}
\int \tan^3(\theta)\sec(\theta)\,d\theta &= 27\int \tan^3(\theta)\sec(\theta)\cdot\frac{1}{\tan(\theta)\sec(\theta)}\,du \\
&= 27\int \tan^2(\theta)\,du \\
&= 27\int (\sec^2(\theta)-1)\,du \\
&= 27\int u^2 - 1\,du = 27(u^3/3 - u) + c \\
&= 27\left(\frac{1}{3}\sec^3(\theta) - \sec(\theta)\right) + c \\
&= 9\sec^3(\theta) - 27\sec(\theta) + c.
\end{aligned}$$

(d) Trigonometry rules give $\tan(\theta) = x/3$ (note this is equivalent to the substitution $x = 3\tan(\theta)$ from part b). Then

Pythagoras' Theorem gives the hypotenuse as $\sqrt{x^2+9}$, so that

$$\cos(\theta) = \frac{3}{\sqrt{x^2+9}}$$

and

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\sqrt{x^2+9}}{3}.$$

(e) Finally, we put together our work from parts b, c and d to solve the integral. From parts b and c, we have

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = 9\sec^3(\theta) - 27\sec(\theta) + c,$$

where $\tan(\theta) = x/3$. This is the situation in our triangle from part d, so we can use our formula for $\sec(\theta)$ to get

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+9}}\,dx &= 9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 27\left(\frac{\sqrt{x^2+9}}{3}\right) + c \\ &= \frac{(\sqrt{x^2+9})^3}{3} - 9\sqrt{x^2+9} + c.\end{aligned}$$