MAT126 Homework 1 Solutions

Problem 1. Define the function $f(x)$ by

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 1 \\ x & \text{if } x > 1. \end{cases}$$

(a) Using a theorem from class, explain how we know that f is integrable on $[0, 3]$.

(b) Evaluate the definite integral $\int_0^3 f(x)\,dx$.

Solution: (a) On page 345 of the textbook, there is a theorem which states that a function $f(x)$ is integrable on $[a, b]$ if it is continuous or if it has only finitely many jump discontinuities. In our case, $f(x)$ has precisely one jump discontinuity (at $x = 1$), so is integrable.

(b)

$$\begin{aligned} \int_0^3 f(x)\,dx &= \int_0^1 x^2 + 1\,dx + \int_1^3 x\,dx \\ &= \left[x^3/3 + x\right]_0^1 + \left[x^2/2\right]_1^3 \\ &= 1/3 + 1 + 9/2 - 1/2 \\ &= 16/3. \end{aligned}$$

Problem 2. Consider the below definite integral:

$$\int_3^5 \frac{1}{x+3}\,dx.$$

(a) Using left endpoints and 4 subintervals, estimate the value of the integral.

(b) Without computing the exact value of the integral, can we say whether our answer to part (a) is an over-estimate or an under-estimate? Why?

Solution: (a) The width of each subinterval will be $(5-3)/2 = 1/2$, so that means we need the value of $f(x)$ for $x = 3, 3.5, 4$ and 4.5 (note we don't need $f(5)$, since we are using left endpoints).

$$\begin{aligned} f(3) &= 1/6 \\ f(3.5) &= 2/13 \\ f(4) &= 1/7 \\ f(4.5) &= 2/15. \end{aligned}$$

Thus, we can estimate the integral as

$$\int_3^5 \frac{1}{x+3}\,dx = \frac{1}{2}\left(\frac{1}{6} + \frac{2}{13} + \frac{1}{7} + \frac{2}{15}\right) = \frac{543}{1820}.$$

(b) Since $f(x) = \frac{1}{x+3}$ is a decreasing function on the interval $[3, 5]$, left endpoints will give an *over-estimate*. This is because, on each subinterval, $f(x)$ decreases as we go right, so takes its largest value at the left endpoint (and this value is what we use as the height of our rectangle that estimates the area under $f(x)$).

Problem 3. You have landed an internship at NASA. It is the day of a big launch, and it is your job to track the height above the earth's surface of the spacecraft. 1 minute after lift-off, all data readings are malfunctioning except for the velocity of the ship at 10 second intervals. Use the below table to estimate the height of the ship after 1 minute.

Time (s)	0	10	20	30	40	50	60
Velocity (m/s)	0	12	40	65	84	72	91

Solution: We could use any of our rules for estimating the area under the graph. I will demonstrate the left endpoint rule here. So, we are going to assume that on each 10 second interval, the velocity stays constantly at the starting velocity of that interval. Graphing this, and calculating the areas of the resulting rectangles gives

$$10(0 + 12 + 40 + 65 + 84 + 72) = 2730.$$

Thus, we estimate the height of the ship to be $2730m$ after 60 seconds.

Problem 4. Let $g(t) = (2t+1)^{1/3}$.

(a) Find an anti-derivative $G(t)$ of $g(t)$.

(b) Evaluate the definite integral

$$\int_{-\frac{1}{2}}^{2} g(t)\,dt.$$

Solution: (a) We know that powers of $2t+1$ will differentiate by decreasing the power by 1, as well as multiplying some nonsense out in front. Thus, to differentiate to $(2t+1)^{1/3}$, we would need to start with some multiple of $(2t+1)^{4/3}$. The chain rule gives that

$$\frac{d}{dt}\left((2t+1)^{4/3}\right) = 2 * \frac{4}{3}(2t+1)^{1/3} = \frac{8}{3}(2t+1)^{1/3},$$

so if we differentiate $\frac{3}{8}(2t+1)^{4/3}$, we will get $g(t)$. Therefore, our anti-derivative is

$$G(t) = \frac{3}{8}(2t+1)^{4/3}.$$

(b) By the Evaluation Theorem (also known as the second part of the Fundamental Theorem of Calculus), we have

$$\begin{aligned}\int_{-\frac{1}{2}}^{2} g(t)\,dt &= G(2) - G\left(-\frac{1}{2}\right)\\ &= \frac{3}{8}(2*2+1)^{4/3} - \frac{3}{8}\left(2*-\frac{1}{2}+1\right)^{4/3}\\ &= \frac{3}{8}\cdot 5^{4/3} - \frac{3}{8}(0)^{4/3}\\ &= \frac{3}{8}\cdot 5^{4/3}.\end{aligned}$$

Problem 5. A function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$ for all x.

(a) Say whether or not each of the following functions are odd:

(i) $f(x) = 3x^3 + 5x$.

(ii) $g(x) = \cos(x + 2)$.

(iii) $h(x) = 2\sin(x)$.

(b) Give another example of an odd function, and draw a graph of your example function on the interval $[-5, 5]$.

(c) Let $f(x)$ be an arbitrary odd function. What is the value of the below definite integral?

$$\int_{-1000000}^{1000000} f(x)\,dx.$$

Solution: (a)(i). $f(-x) = 3(-x)^3 + 5(-x) = -3x^3 - 5x = -f(x)$, so $f(x)$ is **odd**.

(a)(ii). Note that $g(-2) = \cos(0) = 1$. Meanwhile, $g(2) = \cos(4) \neq -1$, since $\cos(x) = -1$ for (every other) multiple of π. Therefore, $-g(2) \neq g(-2)$, and $g(x)$ is **not odd**.

(a)(iii). $h(-x) = 2\sin(-x) = -2\sin(x) = -h(x)$, so $h(x)$ is **odd**.

(b) There are all manner of examples. Any multiple of $\sin(x)$, or any multiple of an odd power of x, or any sum of these functions, are all odd.

(c) The integral is equal to zero. Intuitively, you can hopefully see from your graph in part (b) that odd functions have their areas for positive x and negative x cancelling each other out. We can prove this formally in the following way. First we split the integral up at zero, and then make the substitution $u = -x$

for the first of the two resulting integrals. This, combined with the fact that $f(x)$ is odd so that $f(-u) = -f(u)$, gives our answer (try to following along and understand what we have done at each step):

$$\begin{aligned}
\int_{-1000000}^{1000000} f(x)\,dx &= \int_{-1000000}^{0} f(x)\,dx + \int_{0}^{1000000} f(x)\,dx \\
&= \int_{1000000}^{0} f(-u)(-1)\,du + \int_{0}^{1000000} f(x)\,dx \\
&= \int_{0}^{1000000} f(-u)\,du + \int_{0}^{1000000} f(x)\,dx \\
&= \int_{0}^{1000000} -f(u)\,du + \int_{0}^{1000000} f(x)\,dx \\
&= -\int_{0}^{1000000} f(u)\,du + \int_{0}^{1000000} f(x)\,dx \\
&= -\int_{0}^{1000000} f(x)\,dx + \int_{0}^{1000000} f(x)\,dx \\
&= 0.
\end{aligned}$$

Many people made some variation of the following MISTAKE. Below, F is an anti-derivative of f.

$$\begin{aligned}
\int_{-1000000}^{1000000} f(x)\,dx &= F(1000000) - F(-1000000) \\
&= F(1000000) + F(1000000) = 2F(1000000).
\end{aligned}$$

Can you see why this is wrong?